# Mergers and Ejections of Black Holes in Globular Clusters

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N-Body Simulations

Subsystem Treatment

Post-Newtonian Terms

Full Simulations

Recent Results

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# Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,t + \frac{1}{2}\mathbf{F}_0^{(2)}\,t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\,t^3 \\
\mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\,t + \frac{1}{2}\mathbf{F}_0^{(3)}\,t^2
\end{aligned}$$

Prediction

$$\begin{aligned}
\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}_0^{(1)}\,\delta t'_j + \frac{1}{2}\mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0)\,\delta t'_j + \mathbf{r}_0 \\
\mathbf{v}_j &= ((\frac{1}{2}\mathbf{F}_\mathbf{0}^{(1)}\,\delta t'_j + \mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0\,; \qquad \delta t'_j = t - t_0
\end{aligned}$$

New forces $\qquad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\begin{aligned}
\mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{6}{t^3} \\
\mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{2}{t^2}
\end{aligned}$$

Corrector for $i$

$$\begin{aligned}
\Delta\mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5 \\
\Delta\mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4
\end{aligned}$$

# Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n << \Delta t_d, \qquad n << N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

# Stellar Evolution

| | |
|---|---|
| Stellar HR types | $K^* = 0, ..., 15$ |
| Fast look-up (Pop I & II) | $r^*(t)$, $m_c(t)$, $L^*(t)$, $K^*(t)$ |
| Wind mass loss | $\dot{m} = -2 \times 10^{-13} r^* L^*/m$ |
| Single stars | $\Delta m/m > 1\,\%$, new $r^*$ |
| Updating times | $T_{\rm ev} = t + \min(\Delta t_{\rm ev}, \Delta t_{\rm rem})$ |
| Stellar rotation | $\Delta J_{\rm spin} = 2\Delta m r^2 \Omega_{\rm rot}/3$ |
| White dwarfs | cooling curves, $\Delta t_{\rm ev} = 10^6\,{\rm yr}$ |
| Supernova outburst | $m_{\rm c} > m_{\rm chandra} \Rightarrow$ SN |
| NS velocity kick | $v >> v_\infty \sim 2\,km/s$ |
| Binary mass loss | $m\,a = {\rm const}$ |
| Synthetic HR diagram | binaries and single stars |
| Energy conservation | $\Delta E = \Delta m\,(\frac{1}{2} v^2 + \Phi)$ |

# BH Slingshot

Initial conditions $N = 100\,000$, Kroupa IMF in 0.1–30 $M_\odot$

Plummer & tides, $R_h = 3.0$ pc, $\bar{V} = 7$ km/s

Model 2, t = 3.6 Gyr

Binary: $m_1 = 24$, $m_2 = 14$ M$_\odot$, $a = 20$ AU, $v_\infty = 19$ km/s

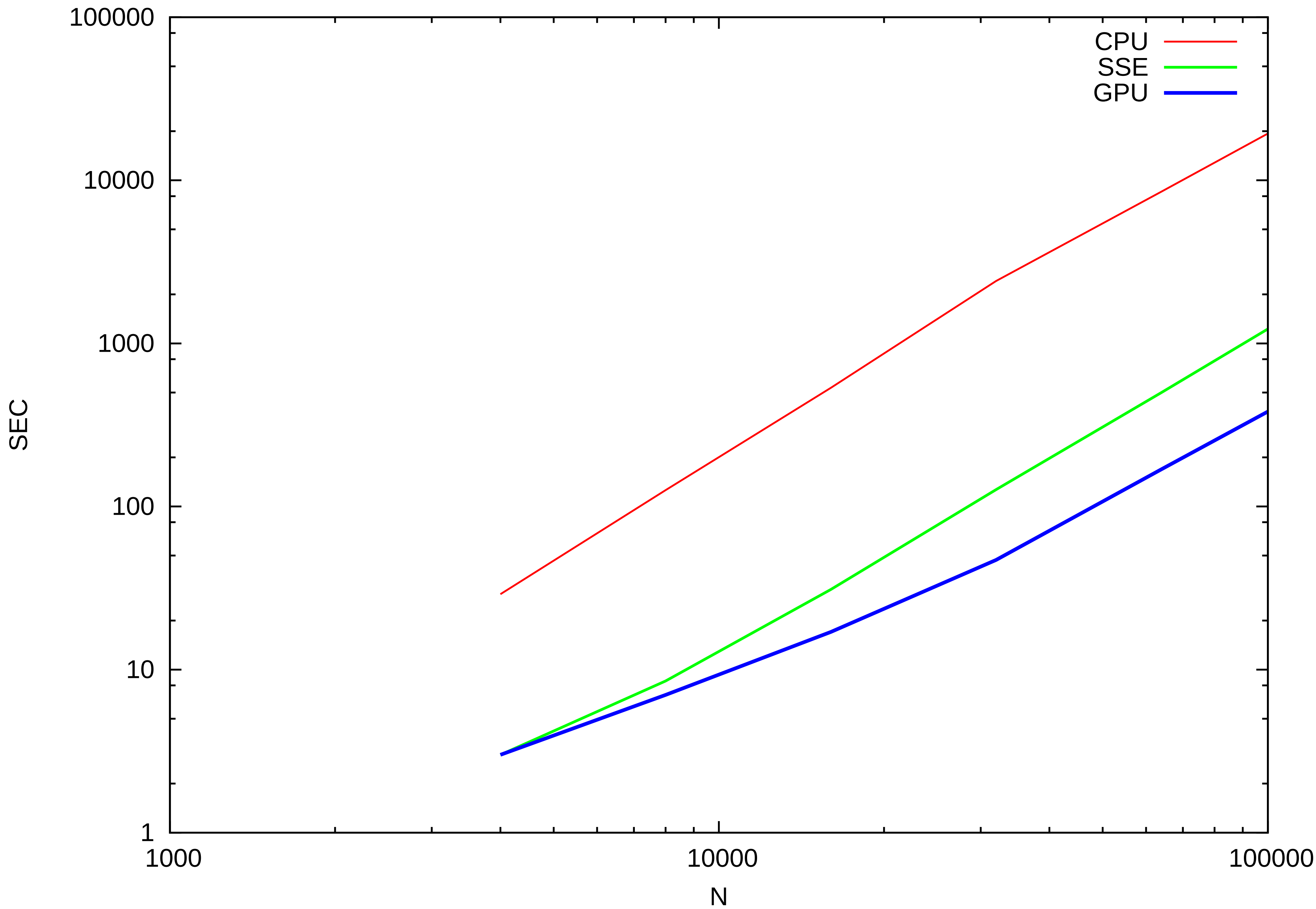
Single: $m_3 = 12$ M$_\odot$, $v_\infty = 69$ km/s

=========================================

Model 4, t = 800 Myr

Binary: $m_1 = 24$, $m_2 = 24$ M$_\odot$, $a = 22$ AU, $v_\infty = 23$ km/s

Single: $m_3 = 10$ M$_\odot$, $v_\infty = 127$ km/s


CPU
SSE
GPU
100000
10000
1000
100
10
1
SEC
1000
10000
100000
N

# Compact Systems

| | |
|---|---|
| Selection procedure | B + S, $a < R_{\rm cl}$ |
| Initialization | c.m. & internal chain |
| Boundaries | subsystem & perturbers |
| Membership updating | ejection or absorption |
| Differential corrections | c.m. chain force |
| Strategy | small subsystem |
| Implementation | ARW-chain + NBODY6 |

NBODY7

# Post-Newtonian Terms

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

GR perturbation

$$\mathbf{P}_{\mathrm{GR}} = \frac{m_1 m_2}{c^2 r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

First-order precession $$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Gravitational radiation

$$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Gradual GR effects $$t_{\mathrm{GR}} < 500,\ 50,\ 1, \quad \mathrm{PN} = 1,\ 2,\ 3$$

Energy check $$E_{\mathrm{tot}} - \int \mathbf{P}_{\mathrm{GR}} \cdot \mathbf{v}\, dt = \mathrm{const}$$

Coalescence $$R < \frac{8\,M}{c^2}, \quad c = \frac{3\times 10^5}{V^*}$$

# PN Elements

Energy $$\epsilon_b = \epsilon_0 + \frac{\epsilon_1}{c^2} + \frac{\epsilon_2}{c^4} + \frac{\epsilon_3}{c^6}, \qquad a = -\frac{M}{2\epsilon_b}$$

$$\epsilon_0 = \frac{1}{2}V^2 - \frac{M}{R}, \qquad \eta = \frac{m_1 m_2}{M^2}$$

$$\epsilon_1 = \frac{1}{2}\left(\frac{M}{R}\right)^2 + \frac{3}{8}(1-3\eta)V^4 + \frac{1}{2}\left((3+\eta)V^2 + \eta\dot{R}^2\right)\frac{M}{R}$$

Lenz vector $$\mathbf{e} = \mathbf{V} \times \mathbf{R} \times \mathbf{V}/M - \mathbf{R}/R$$

Periapse advance $$\Delta\omega = \frac{6\pi M}{c^2 a(1-e^2)}$$

PN2.5 $$\tau_{GR} = \frac{5g(e)}{64}\frac{a^4 c^5}{X(1+X)m_1^3}, \qquad X = \frac{m_2}{m_1}$$

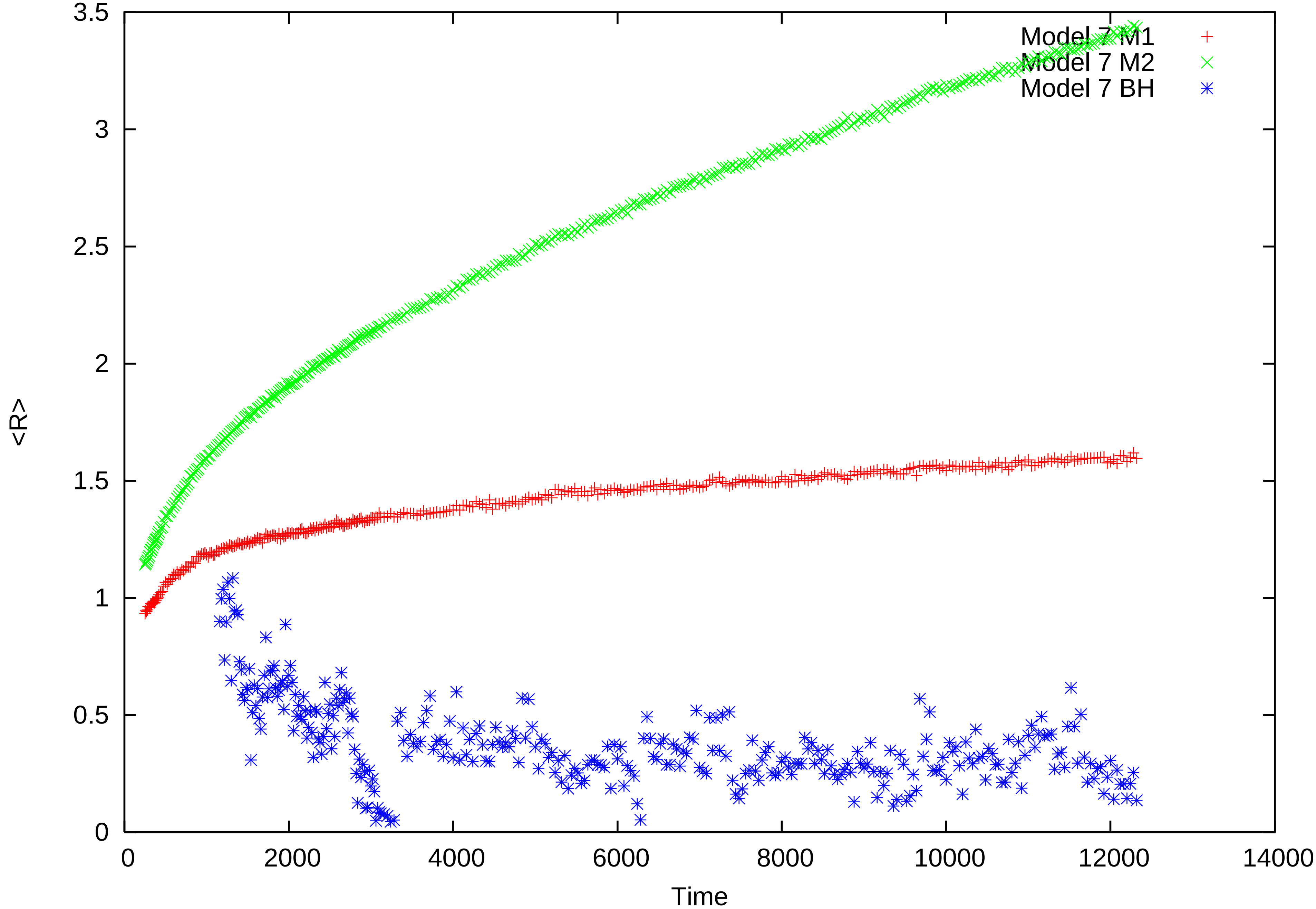
$$g(e) \simeq \frac{(1-e^2)^{7/2}}{4.5}$$

Angular momentum $$\mathbf{J} = \mathbf{J}_0(1 + f_1/c^2 + f_2/c^4)$$

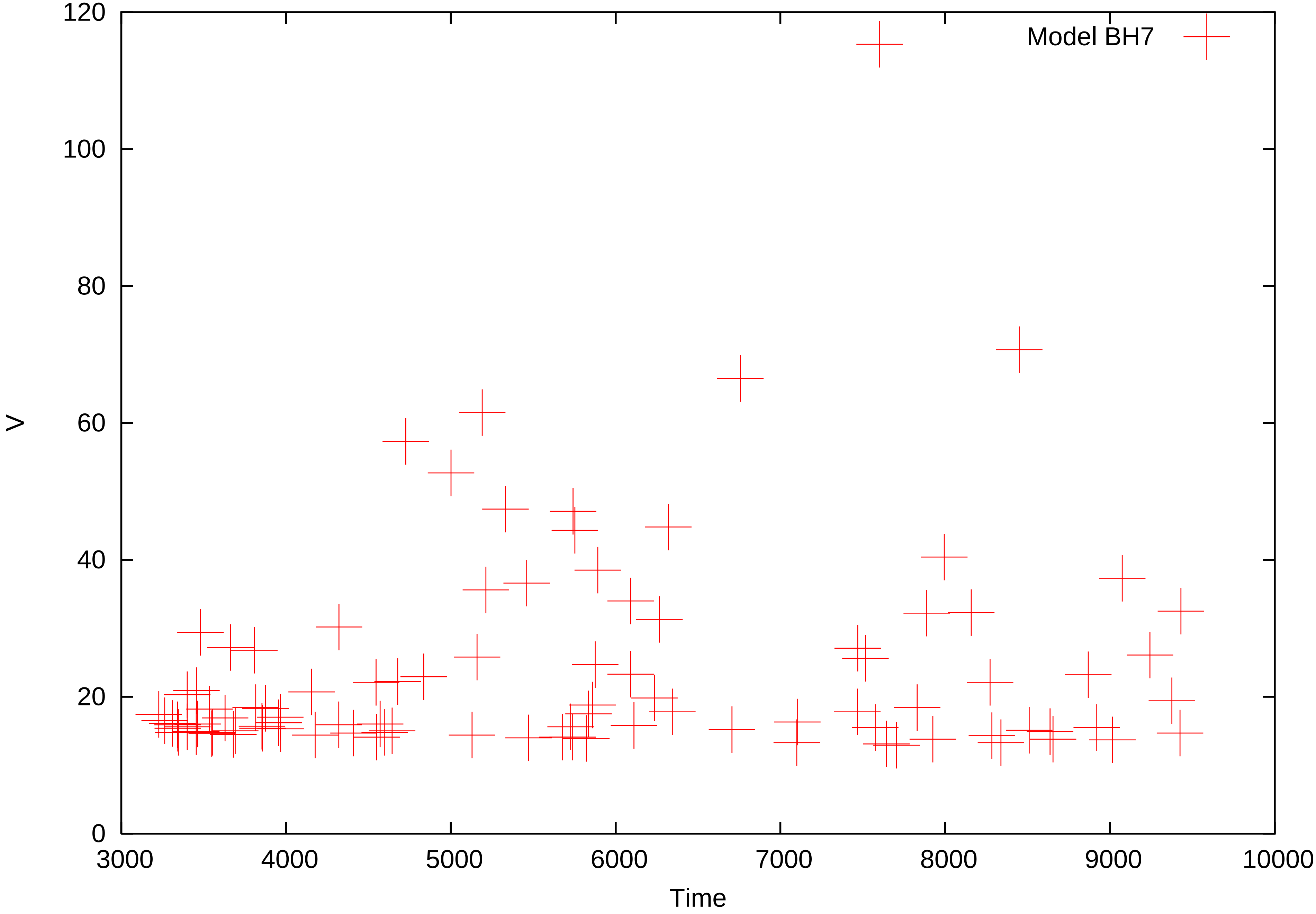
Eccentricity $$e^2 = (1 - \frac{\mathbf{J}^2}{Ma})$$

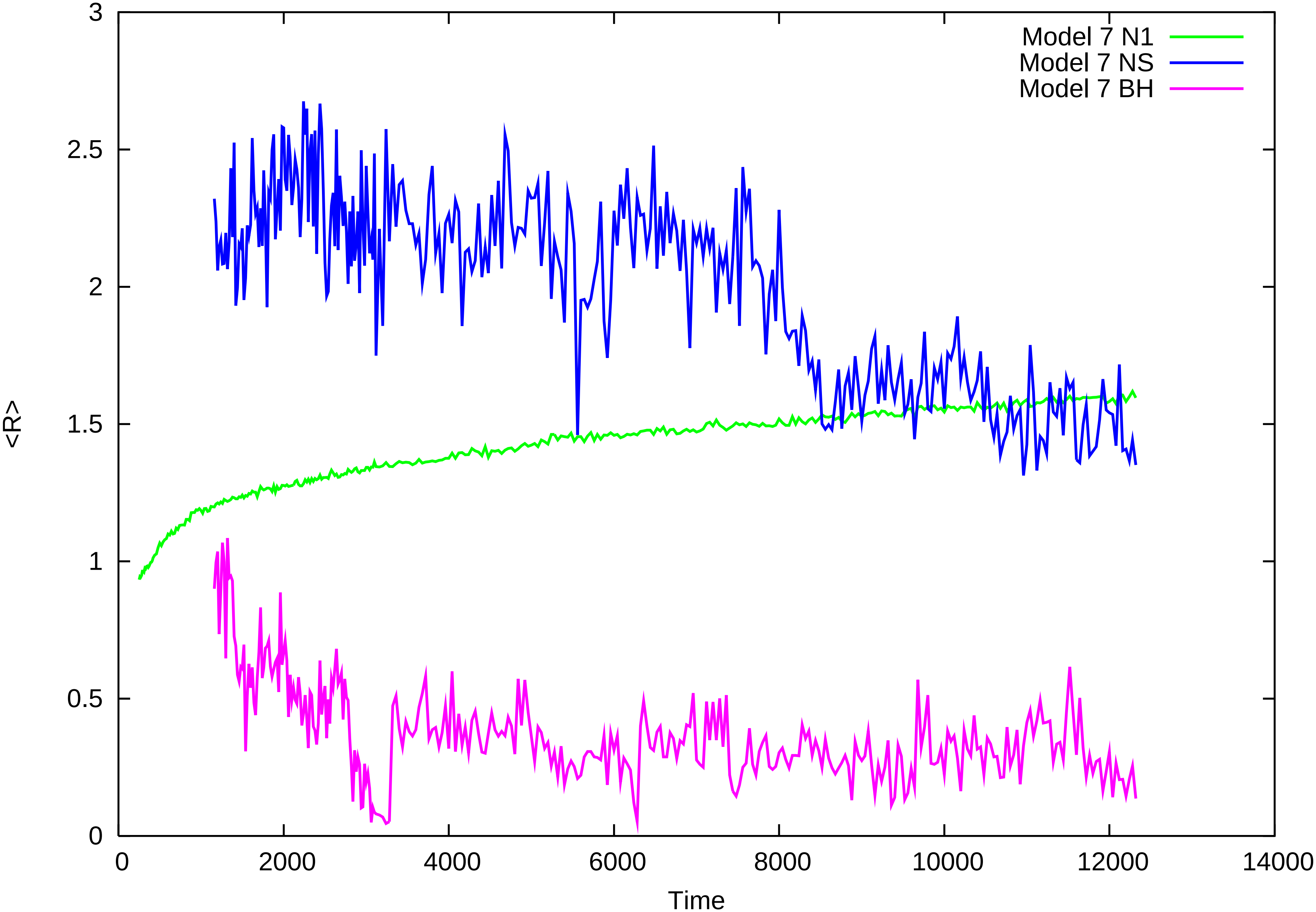
# Full Simulations

| | |
|---|---|
| Initial conditions | $N = 100\,000,$ Kroupa IMF in 0.1–50 $M_\odot$ |
| | Plummer & tides, $R_h = 1.0$ pc, $V^* = 16$ km/s |
| Stellar evolution | BH/NS formation above 20/8 $M_\odot$, # 150/680 |
| Velocity kicks | Maxwellian $\sigma = 2V^*, \quad v_\infty \simeq (1-2)V^*$ |
| N-body units | $G = 1, \; \bar{m} = 1/N, \; E = -\frac{1}{4}, \quad \Rightarrow \bar{v}^2 = \frac{1}{2}, \; R_{\rm h} \simeq 1$ |
| Hard binary | $\frac{m^2}{2a} \simeq \frac{1}{2}\bar{m}\bar{v}^2, \quad \Rightarrow a_{\rm hard} \simeq 2/N$ |
| Hard limit | $-\frac{m^2}{2a} = E, \quad \Rightarrow a_0 = 2/N^2,$ or $2 \times 10^{-10}$ here |
| Dynamics | $m_{\rm bh} \simeq 20\ M_\odot, \quad E_{\rm b} = 0.01E, \quad \Rightarrow a \simeq 2 \times 10^{-5}$ |
| Relativity | $R_{\rm coal} = \frac{8(m_1 + m_2)}{c^2}, \; c = \frac{3 \times 10^5}{V^*}, \quad \Rightarrow R_{\rm coal} \simeq 2 \times 10^{-11}$ |
| PN condition | $a(1-e) \simeq 1 \times 10^3 R_{\rm coal}, \quad \Rightarrow e > 0.999$ |
| Models | $c = 18\,000, \quad R_{\rm coal} \simeq 1 \times 10^{-11}, \quad \tau_{\rm GR} < 500 \quad \simeq 30$ Myr |
| Time-scale | $a = 2 \times 10^{-5}, \; e = 0.999, \; m = 20\ {\rm M}_\odot, \; \Rightarrow \tau_{\rm GR} \simeq 2$ Myr |

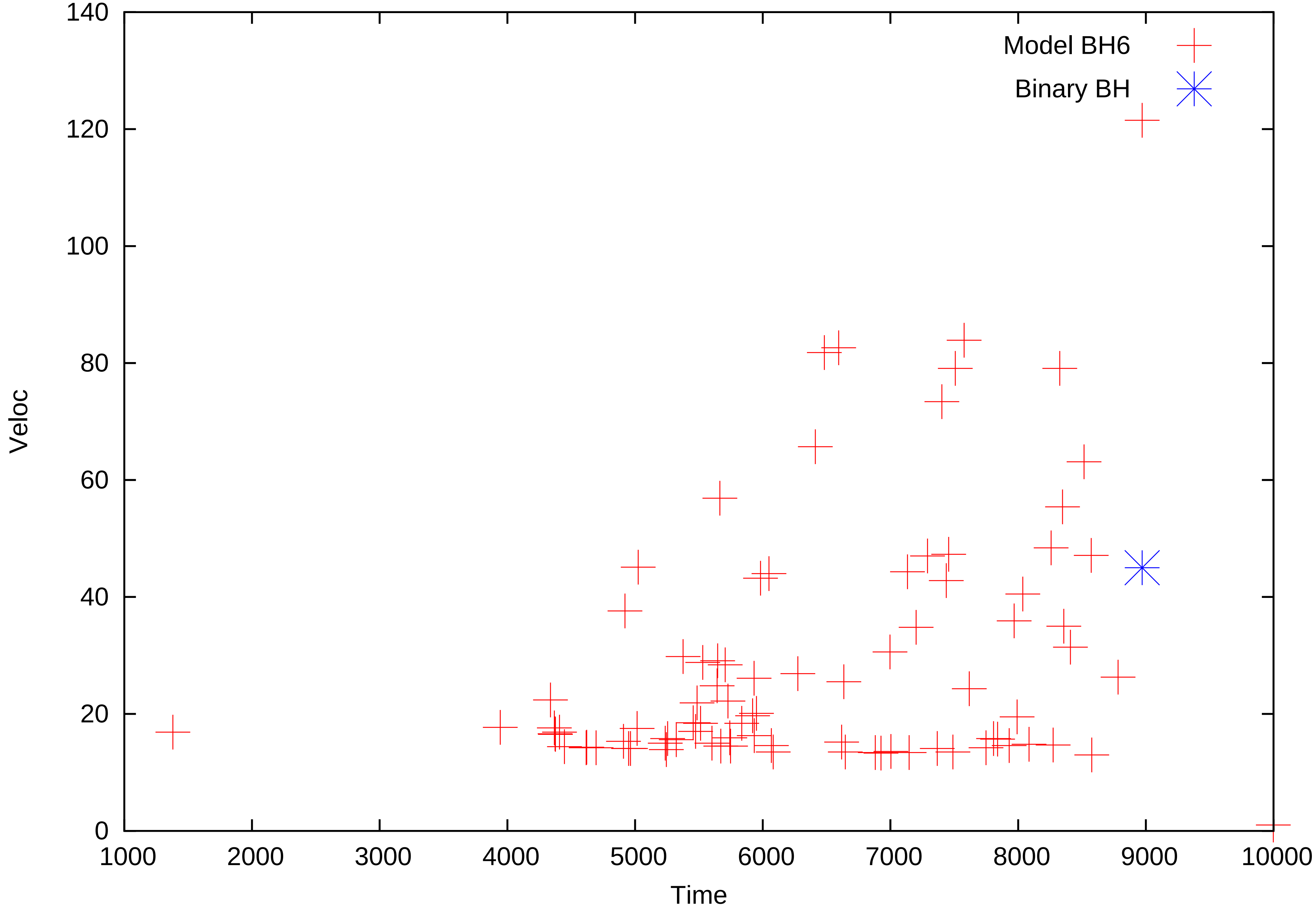


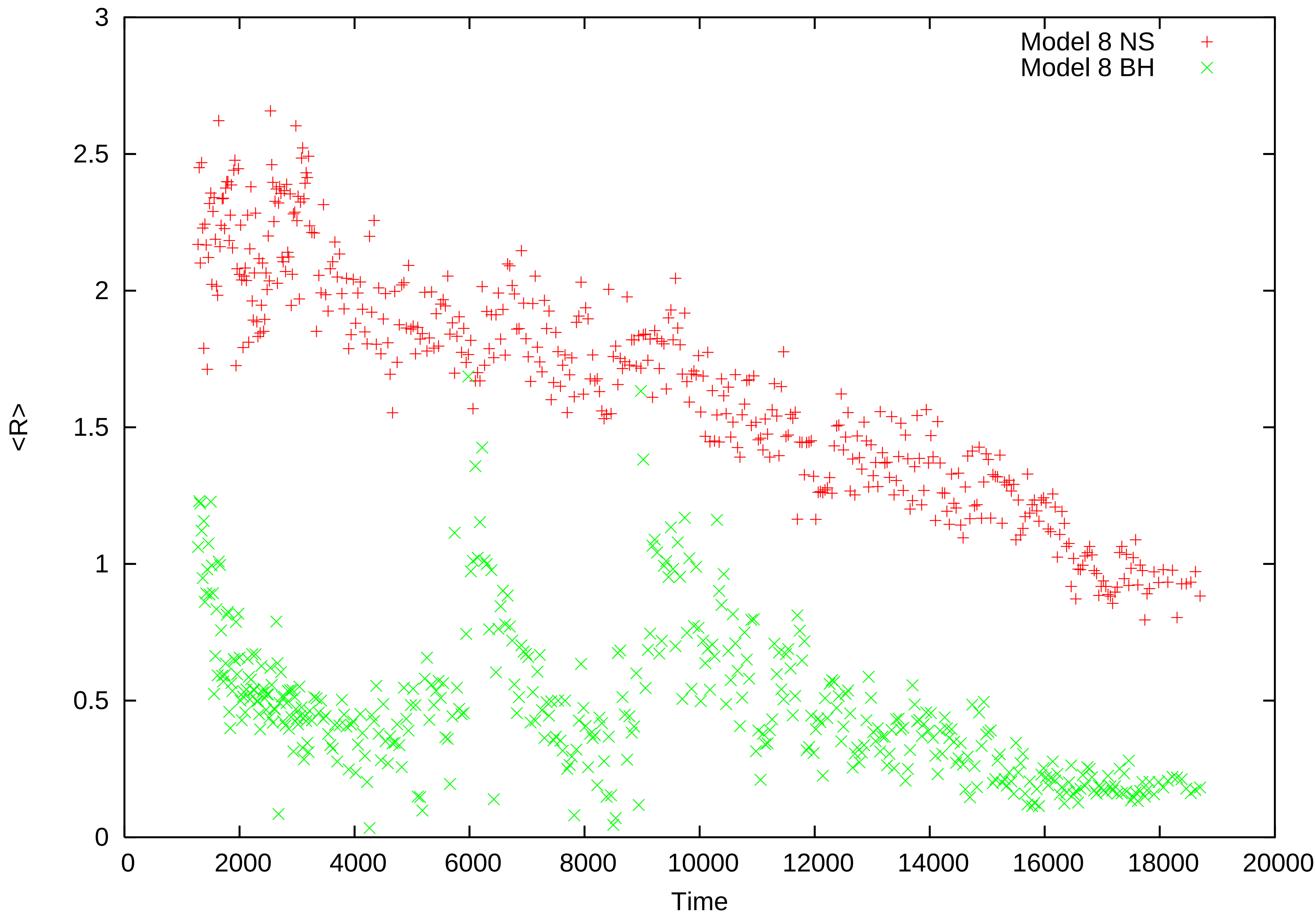
Model 7 M1
Model 7 M2
Model 7 BH
<R>
Time
0
0.5
1
1.5
2
2.5
3
3.5
0
2000
4000
6000
8000
10000
12000
14000


Model BH7
120
100
80
60
40
20
0
V
3000
4000
5000
6000
7000
8000
9000
10000
Time

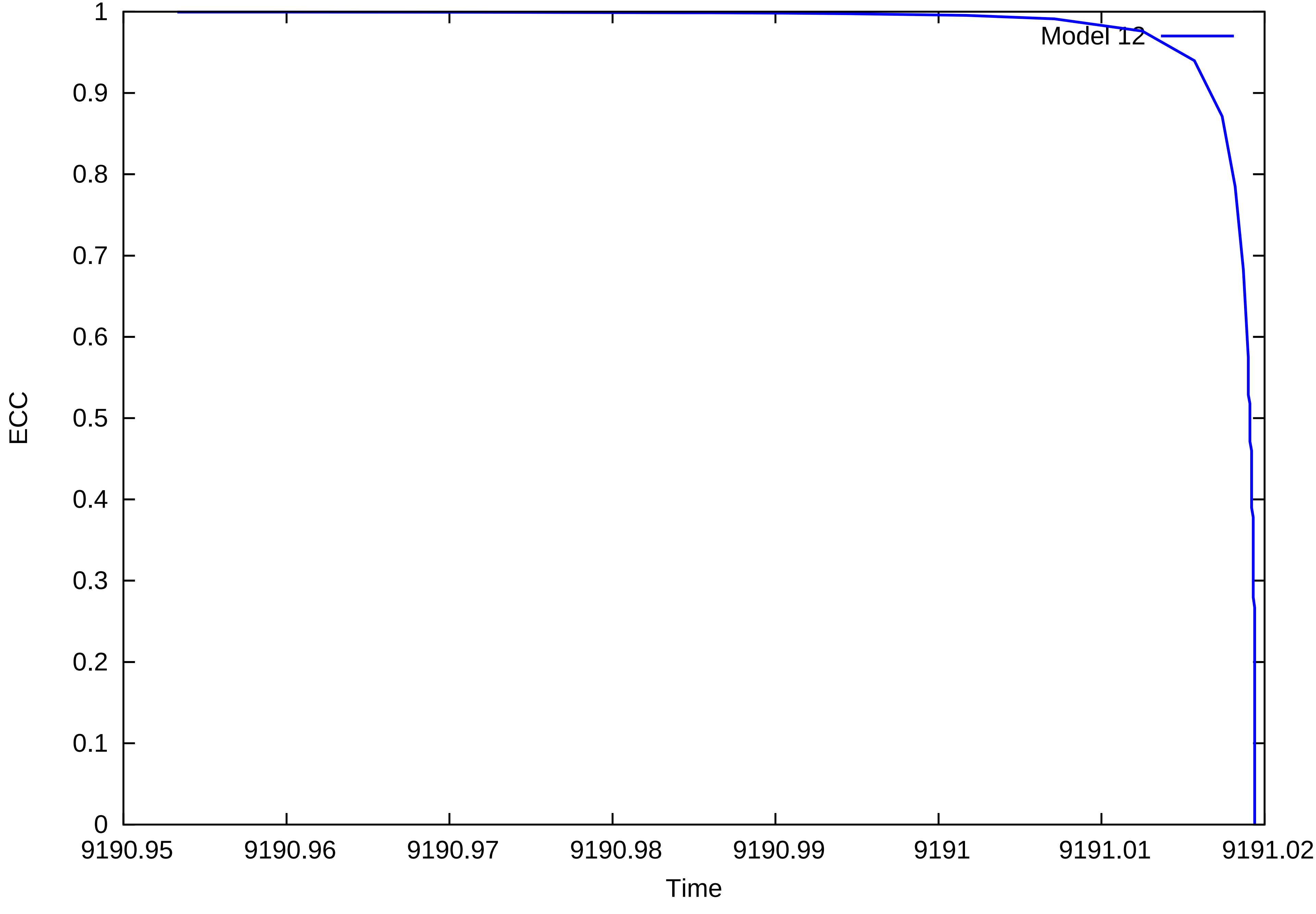


Model 7 N1
Model 7 NS
Model 7 BH
<R>
Time
0
0.5
1
1.5
2
2.5
3
0
2000
4000
6000
8000
10000
12000
14000

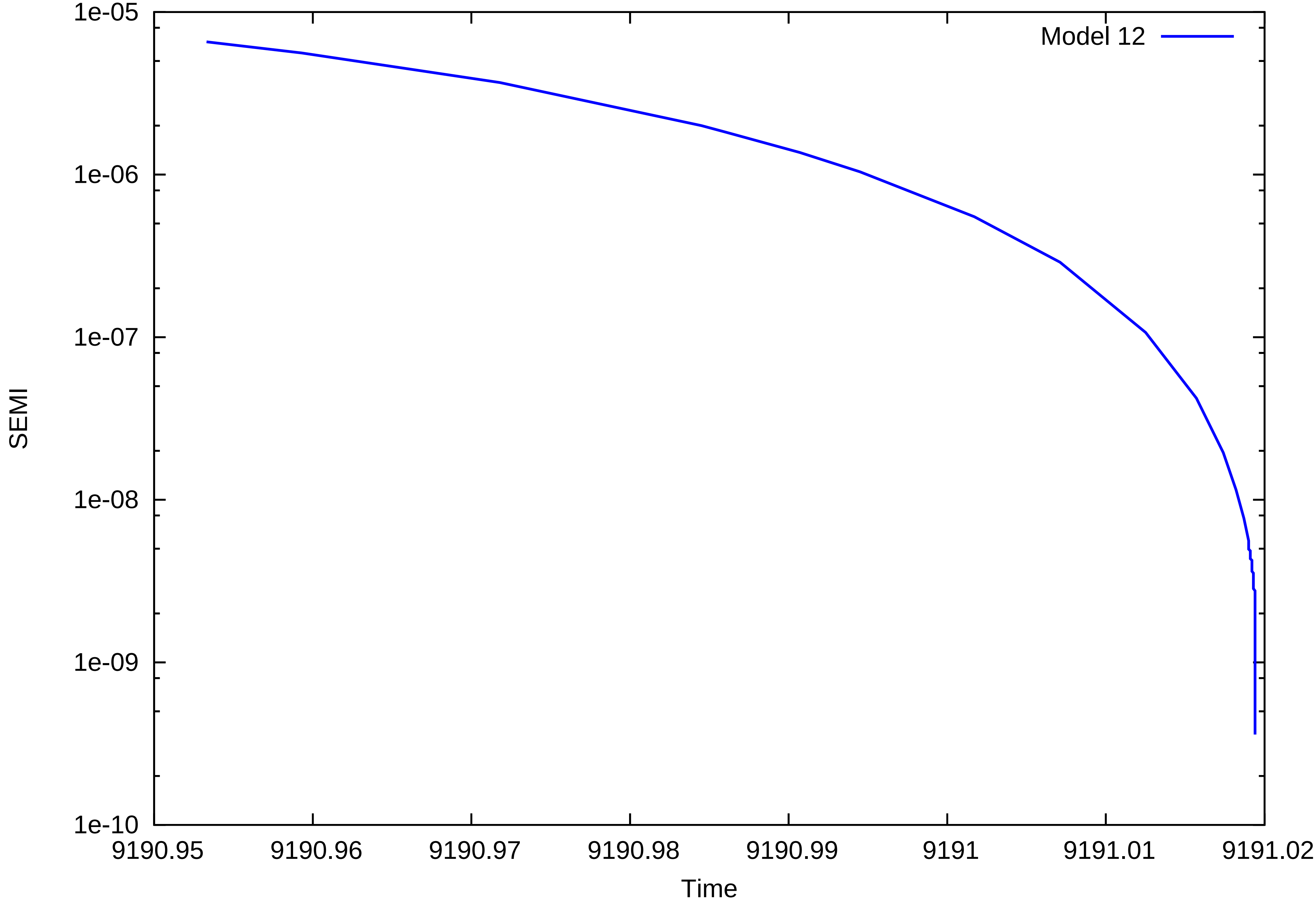

Model BH6
Binary BH
Veloc
Time
0
20
40
60
80
100
120
140
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000



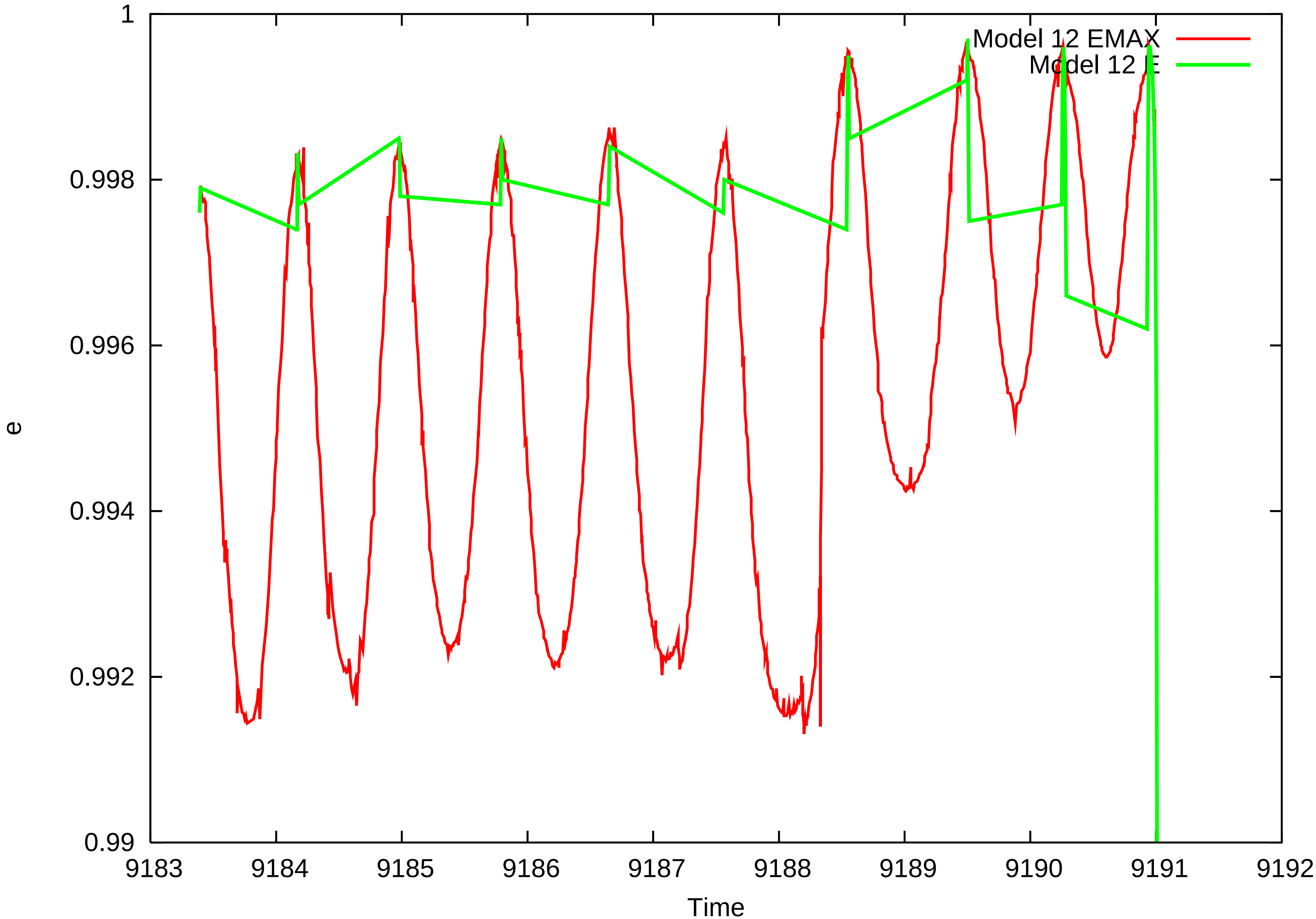
Model 8 NS
Model 8 BH
<R>
Time
0
0.5
1
1.5
2
2.5
3
0
2000
4000
6000
8000
10000
12000
14000
16000
18000
20000



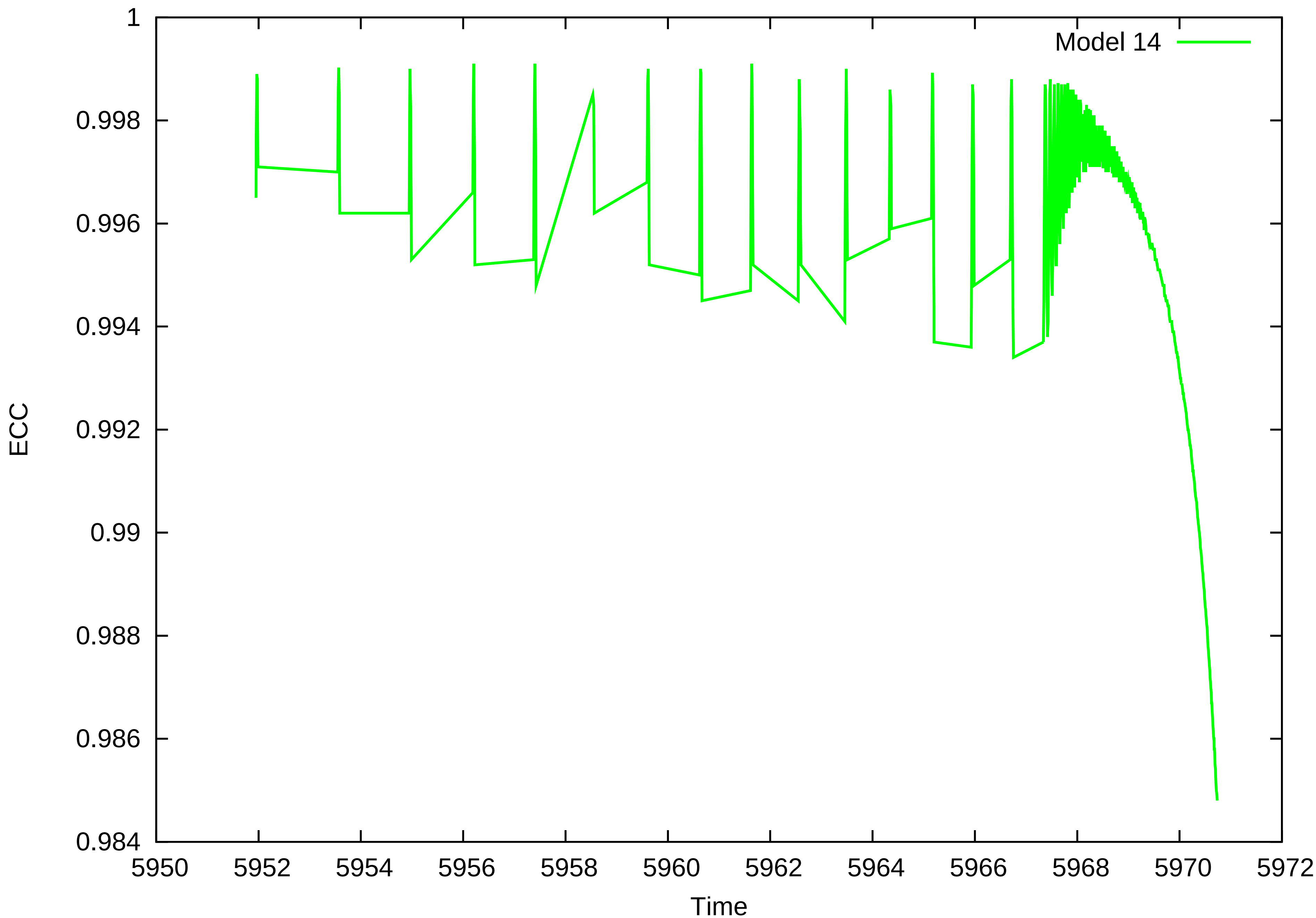
Model 12
ECC
Time
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
9190.95
9190.96
9190.97
9190.98
9190.99
9191
9191.01
9191.02

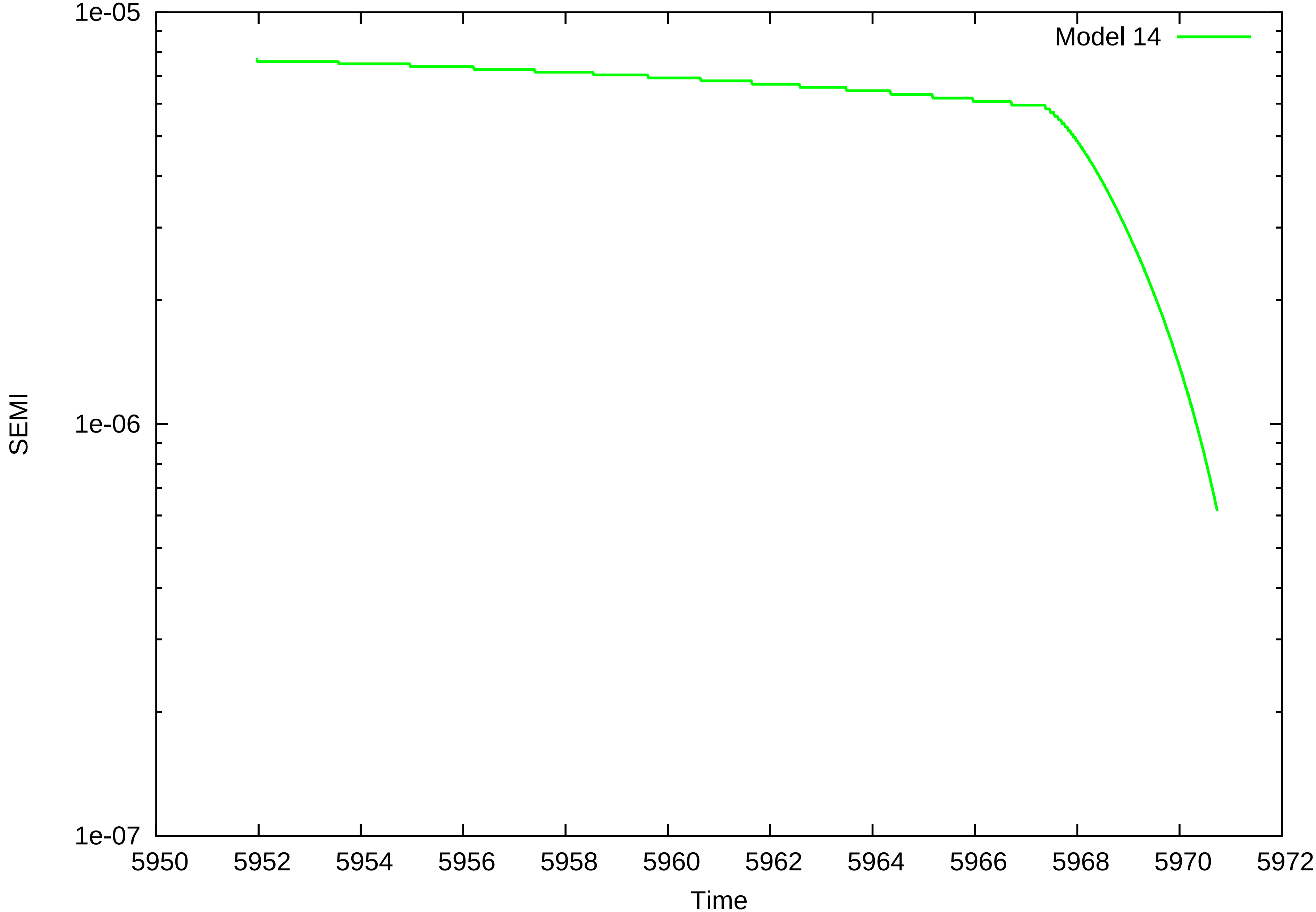


Model 12
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
SEMI
9190.95
9190.96
9190.97
9190.98
9190.99
9191
9191.01
9191.02
Time



Model 12 EMAX
Model 12 E
1
0.998
0.996
0.994
0.992
0.99
e
9183
9184
9185
9186
9187
9188
9189
9190
9191
9192
Time


Model 14
ECC
Time
1
0.998
0.996
0.994
0.992
0.99
0.988
0.986
0.984
5950
5952
5954
5956
5958
5960
5962
5964
5966
5968
5970
5972



Model 14
SEMI
Time
1e-05
1e-06
1e-07
5950
5952
5954
5956
5958
5960
5962
5964
5966
5968
5970
5972

# Key Stages Model BH12

BH binary $t = 465, \quad N_{\rm bh} = 9, \quad a = 7 \times 10^{-5}$

Subsystem $t = 533, \quad a = 2 \times 10^{-5}, \quad R_{\rm s} = 2 \times 10^{-1}$

Kozai cycles $t = 552, \quad i = 98, \quad T_{\rm koz} = 0.1$ Myr

Eccentricity $t = 552, \quad a = 1 \times 10^{-5}, \quad e = 0.997$

Kozai cycles $t = 553, \quad e_{\rm max} = 0.9996, \quad e = 0.9996$

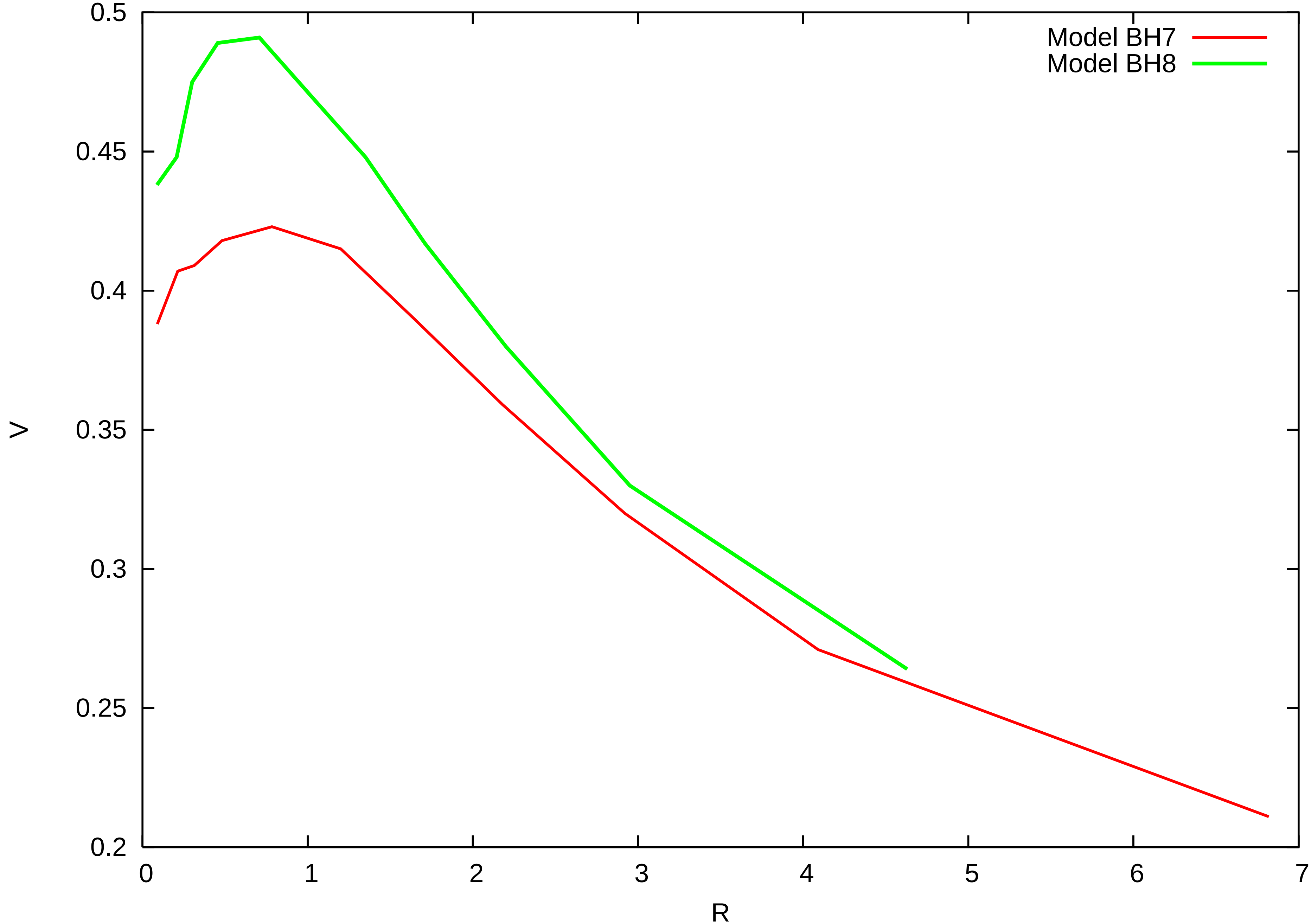
Coalescence $t = 554, \quad a = 1 \times 10^{-11}, \quad e < 0.03$

Eccentricity $t = 564, \quad a = 3 \times 10^{-4}, \quad e = 0.9999$

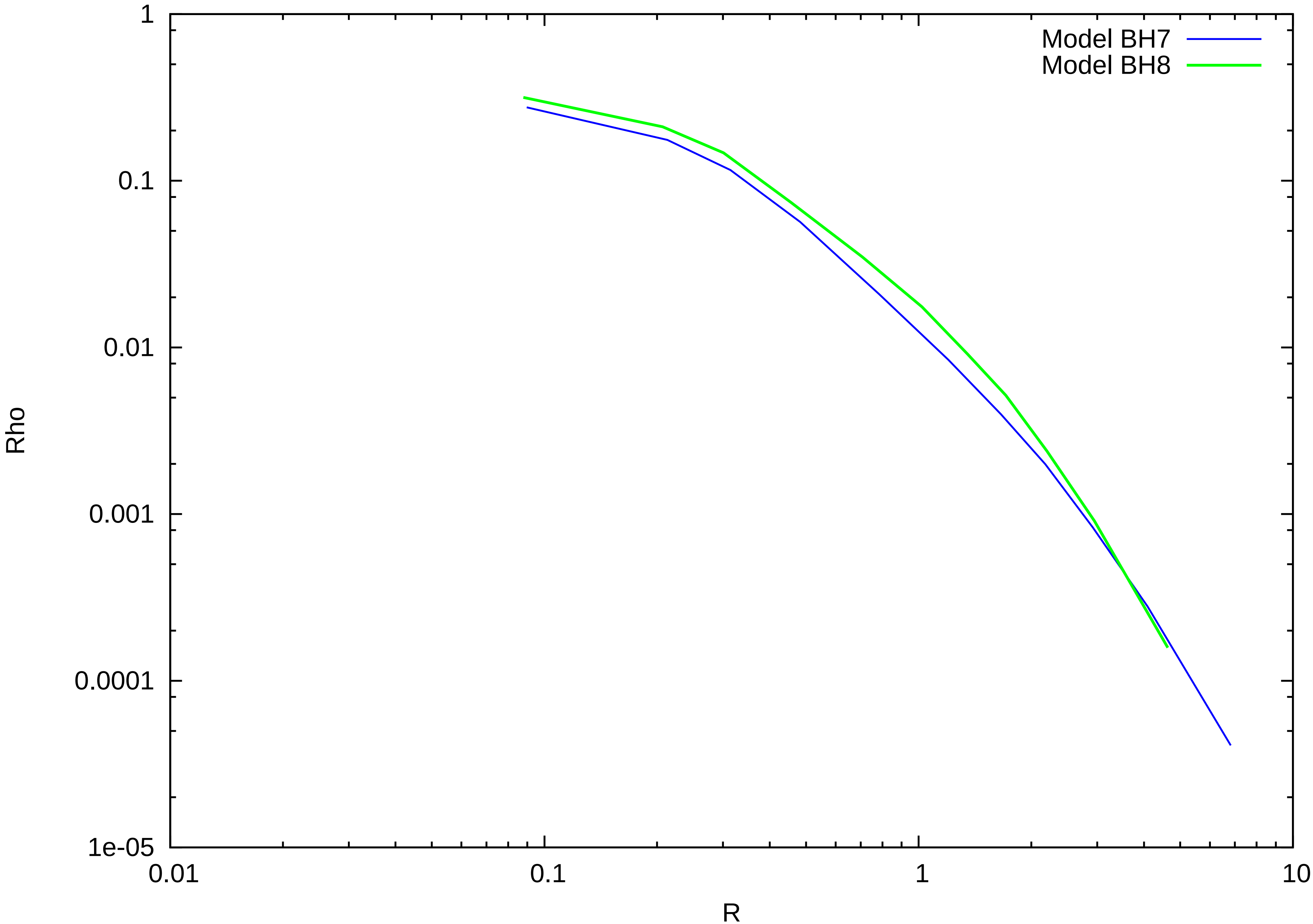
Shrinkage $t = 638, \quad a = 3 \times 10^{-5}, \quad e = 0.67$

Subsystem $t = 718,\ a = 2{\times}10^{-5},\ e = 0.999,\ R = 10^{-3}$

Coalescence $t = 720, \quad a = 4 \times 10^{-8},$ unperturbed



Model BH7
Model BH8
V
R
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0
1
2
3
4
5
6
7



Model BH7
Model BH8
Rho
1
0.1
0.01
0.001
0.0001
1e-05
0.01
0.1
1
10
R

# No BH Velocity Kicks

| | |
|---|---|
| Current epoch | $t = 5 \times 10^4, \quad T_{\mathrm{phys}} = 3.0$ Gyr |
| Initial BH populations | $N_{\mathrm{bh}} = 147$ |
| Latest BH population | $N_{\mathrm{bh}} = 39$ |
| BH binary escapers | $NB_{\mathrm{esc}} = 11$ |
| GR coalescence | $N_{\mathrm{coal}} = 1$ |
| Standard binaries | $N_{\mathrm{bin}} = 10$ |
| Stellar escapers | $N^*_{\mathrm{esc}} = 17,000$ |
| Accumulated error | $\sum \Delta E_i = -2 \times 10^{-5}$ |